IX Conference Mathematical Modelling in Physics and Engineering

SOLUTIONS OF SOME FUNCTIONAL EQUATIONS IN A CLASS OF GENERALIZED HÖLDER FUNCTIONS

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The existence and uniqueness of solutions a nonlinear iterative functional equation
in the class of r-times differentiable functions with the r-derivative satisfying a generalized Hölder condition is considered

The space W_γ and its properties

Consider non-linear functional equation

$$\varphi(x) = h(\varphi[f(x)]) + g(x) \qquad (1)$$

where f, g, h are given and φ is a unknown function.
We accept the following notation: $I = [a, b], a, b \in R, d := b - a$, $W_\gamma(I)$ - is the Banach space of the r-time differentiable functions defined on the interval I with values in R, such that, for some M>0 ; its r-th derivative satisfies the following γ-*Hölder condition*

$$\left|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})\right| \leq M\gamma(|x - \bar{x}|), \qquad \bar{x}, x \in \mathrm{I}.$$

where a fixed function γ satisfies the following condition:
(Γ) $\gamma: [0, d] \to [0, \infty)$ is increasing and concave, $\gamma(0) = 0$, $\lim_{t\to 0^+} \gamma(t) = \gamma(0)$, $\lim_{t \to d^-} \gamma(t) = \gamma(d)$, $\gamma'_+(0) = +\infty$

We assume that:

(i) $f: I \to I$, $f \epsilon W_\gamma(I)$, $\sup_I |\mathrm{f}'| \leq 1$

(ii) $g: I \to R$, $g \epsilon W_\gamma(I)$

(iii) $h: R \to R$, $h \in C^r$, $h^{(r)}$ fulfils the Lipschitz condition in R.

(iv) there exists $\xi \in$ I such that $\lim_{n\to\infty} f^n(x) = \xi$, $x \in$ I, where f^n is the n-th iteration function f

(v) h is analityc function at η_0, where η_0 is the solution of equation $\eta_0 = h(\eta_0) + g(\xi)$

We define functions $h_k: I \times R^{k+1} \to R, k = 0,1, \dots, r-1$ by the formula

$$\begin{cases} h_0(x, y_0) := h(y_0) + g(x) \\ h_{k+1}(x, y_0, \dots, y_{k+1}) := \frac{\partial h_k}{\partial x} + f'(x)\left(\frac{\partial h_k}{\partial y_0} y_1 + \dots + \frac{\partial h_k}{\partial y_k} y_{k+1}\right). \end{cases} \qquad (2)$$

If f, g, h satisfy the assumptions (i)-(iii) and $\varphi \epsilon W_\gamma(I)$ is a solution of equation (1) then the derivatives $\varphi^{(k)}, k = 0, \dots, r$ satisfy the system of equations

$$\varphi^{(k)}(x) = h_k\left(x, \varphi[f(x)], \dots, \varphi^{(k)}[f(x)]\right), \qquad x \in I.$$

If assumptions (i)-(iv) are fulfilled and $\varphi \epsilon W_\gamma(I)$ is a solution of equation (1) in I, then the numbers

$$\eta_k = \varphi^{(k)}(\xi), \; k = 0, \dots, r$$

satisfy the system of equations

$$\eta_k = h_k(\xi, \eta_0, \dots, \eta_k), \qquad k = 0, \dots, r$$

where h_k are defined by (2).

Main result

Theorem

If the assumptions (i)-((iii) are fulfilled, f is a function monotone in the interval I, the conditions (iv) and (v) are fulfilled for $\xi = 0$, $\eta_0 = 0$ and

$$h_k(0, \dots, 0) = 0, \quad k = 1, \dots, r$$
$$|h'(0)(f'(0))^r| < 1$$

then the equation (1) has exactly one solution $\varphi \epsilon W_\gamma(I)$ satisfying the condition

$$\varphi^{(k)}(0) = 0, \qquad k = 0, \dots, r \qquad (3)$$

Moreover, there exists a neighbourhood U of the point $\xi = 0$ and the number r_0 such that for a function $\varphi_0 \epsilon W_\gamma(\bar{U})$, satisfying the condition (3) and the inequality $\|\varphi_0\| \leq r_0$, a sequence of functions

$$\varphi_n(x) = h(\varphi_{n-1}[f(x)]) + g(x), \qquad x \in \bar{U},$$

converges to a solution of the equation (1) according to the norm in the function space $W_\gamma(\bar{U})$.

Ministerstwo Nauki i Szkolnictwa Wyższego

Organizacja IX Konferencji Modelowanie Matematyczne w Fizyce i Technice (MMFT 2017)
- zadanie finansowane w ramach umowy 829/P-DUN/2017 ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych na działalność upowszechniającą naukę.

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